A problem demonstrating solution of stiff ODE using the explicit Euler procedure
ref: Numerical methods for Engineers and Scientists - J. Hoffman, pg. 298

In a single ODE “stiffness” is associated with equations which have more than one time scale of interest. The equation below was considered by Gear, 1971

Solve the "stiff" differential Equation: \( y' = -1000(y-(t + 2))+1 \) using the Explicit Euler Method

\[
y(0) = 1 \quad \text{initial condition, value of } y \text{ at time zero is 1}
\]

Using a Taylor series, the following expression can be developed for the forward difference derivative.

\[
y'_n = \frac{y_{n+1} - y_n}{\Delta t} - \frac{1}{2} y''(\tau) \cdot \Delta t = f_n \quad \text{in this case } f_n = -y
\]

Drop the remainder term, \( \frac{1}{2} y''(\tau) \cdot \Delta t \) and solve for \( y_{n+1} \) (an approximation because remainder term was dropped)

\[
y_{n+1} = y_n + \Delta t \cdot f_n = y_n + \Delta t \cdot [-1000 \cdot [y - (t_n + 2)] + 1]
\]

\[
t_{n+1} = t_n + \Delta t = n \cdot \Delta t
\]

This equation has 2 time scales of interest, one very small, the other long. Below is a plot of the solution for very short time frames.

\[
n := 0 \ldots 100 \quad \Delta t_0 := .0000 \quad y_0 := 1
\]

\[
\Delta t := .0002 \quad \Delta t_n := \Delta t
\]

\[
y_{n+1} = y_n + \Delta t_n \cdot [-1000 \cdot [y - (n \cdot \Delta t) + 2] + 1]
\]

Fig. 1. Stiff ODE at short time scale

- time step = 0.0002, 100 time steps, stable
Next we have the same solution for a "long" time frame. Note that the time step used is still very small. This is because if we use one much larger the solution technique becomes unstable.

\begin{align*}
n &:= 0..100000 \quad \Delta t_0 := .0000 \quad y_0 := 1 \\
\Delta t_t &:= .0002 \quad \Delta t_n := \Delta t_t \\
y_{n+1} &:= y_n + \Delta t_n \cdot \left[ -1000 \cdot (y_n - (n \cdot \Delta t_t) + 2) + 1 \right]
\end{align*}

![Fig. 2 Stiff ODE at longer time scale](image)

Below is the same solution using a larger time step. The procedure becomes unstable...as shown below

\begin{align*}
n &:= 0..1000 \quad \Delta t_0 := .0000 \quad y_0 := 1 \\
\Delta t_t &:= .00209 \quad \Delta t_n := \Delta t_t \\
y_{n+1} &:= y_n + \Delta t_n \cdot \left[ -1000 \cdot (y_n - (n \cdot \Delta t_t) + 2) + 1 \right]
\end{align*}
Fig. 3 Stiff ODE unstable solution

time step = 0.00202, 1000 time steps, unstable