Example problem and study sheet. If you study this document carefully you should:

1. Be able to see how the governing differential for acceleration was developed
2. How it was solved, both analytically and numerically
3. The similarities and differences between the 2 solution techniques
4. Why some type of software like Mathcad is useful in doing dynamics

Every attempt has been made to produce a document that is understandable to someone possessing only minimal knowledge of Mathcad. If you do not understand any of the Mathcad statements seek help from someone immediately.

**PROBLEM STATEMENT**

The acceleration of a particle falling through the atmosphere is determined by the relationship

\[ a = g(1 - k^2 v^2). \]

Knowing the particle starts at time \( t = 0 \) sec with no initial velocity, \( v_0 = 0 \):

a. Show that the velocity at time \( t \) is given by:

\[ v = \frac{1}{k} \tanh (k \cdot g \cdot t) \]

b. Plot the velocity of the particle for any value of time \( t \).

c. Why is \( v \) at large \( t = 1/k \) called the terminal velocity?

The problem gives an expression for acceleration. We can see where this expression came from by writing Newton's second law for the particle assuming that the viscous friction term has the form:

\[ F_{viscous} = m \cdot g \cdot v^2 \cdot k^2 \quad \text{force due to drag on particle} \]

\[ F_{weight} = m \cdot g \quad \text{force due to weight of particle} \]

\[ \sum_{i=0}^{i} (F_y = m \cdot a) \]

substituting the force expressions into the 2nd Law:

\[ m \cdot g - m \cdot g \cdot k^2 \cdot v^2 = m \cdot a \]
Solve for the resulting acceleration of the particle: \( a = g (1 - k^2 v^2) \). Note that if the 2 forces are equal the acceleration is zero and the particle moves with a constant velocity.

**SOLUTION**

Drawing imported from Paintbrush, a Windows utility, can be edited in place

If the 2 forces are equal, \( a = ? \)

Rewrite the acceleration expression as a first-order, non-linear, differential equation

\[
\frac{dv}{dt} = a = g (1 - k^2 v^2)
\]

Group like terms and integrate:

\[
\int_0^t g \, dt = \int_0^v \frac{1}{g (1 - k^2 v^2)} \, dv
\]

By integration using Mathcad, right side yields:

\[
\frac{-1}{2} \frac{(\ln(k \cdot v - 1) - \ln(k \cdot v + 1))}{k}
\]

**NOTE:** with symbolic integration using Maple, constant of integration not included automatically has to be added manually

we get, upon integration of both sides: \( g \cdot t = \frac{-1}{2} \frac{(\ln(k \cdot v - 1) - \ln(k \cdot v + 1))}{k} + C \)
solve for velocity, using symbolic solve in Mathcad, to get: 
\[ v = \frac{-\left(1 + \exp(-2C \cdot k + 2g \cdot t \cdot k)\right)}{\left(k - \exp(-2C \cdot k + 2g \cdot t \cdot k)\right) \cdot k} \]

In order to obtain a complete solution we must solve for the constant of integration, C, with known values of time and velocity; using the symbolic “solve” in Mathcad we can solve the expression above for C

\[ C = \frac{-1}{2} \ln\left(\frac{-\left(\frac{k \cdot v + 1}{1 - k \cdot v}\right)}{2g \cdot t \cdot k}\right) \]

To evaluate the constant of integration C, we know that at \( t = 0 \) sec, \( v = 0 \) m/sec

\[ C = \frac{-1}{2} \ln\left(\frac{-\left(1 / 1\right)}{2g \cdot t \cdot k}\right) \]

The argument of the \( \ln \) function reduces to -1 and the entire expression reduces to:

\[ C(k) = \frac{-1}{2} \ln(-1) - \frac{2 \cdot g \cdot t \cdot k}{k} \]. By examination we see that, \( C = \frac{-1}{2} \cdot \ln(-1) \), when \( v = 0 \) and \( t = 0 \).

Beer and Johnson report, for the same integration:

\[ v_2 = \tanh(k \cdot g \cdot t) \cdot \frac{1}{k} \]

At this point the user should take note that a symbolic manipulator in software applications is often unpredictable (at least to the uninitiated user) in the answer it gives; not that the answers are wrong they simply look different than expected.

We are confronted by two expressions which look quite different in appearance. A practical approach to determine whether or not they provide the same result is to plot them and see if both equations give the same value for a range of times from .1 to 5 seconds, evaluated every .02 seconds, (245 points).

assign a value to k for evaluation purposes: \( k := .1 \cdot \text{sec} \cdot \text{m} \)
\[ t := .1 \text{ sec}, .12 \text{ sec}, 5.00 \text{ sec} \quad \text{range variable for plot} \]

\[ v_{\text{anal2}}(t) := \tanh(k \cdot g \cdot t) \cdot \frac{1}{k} \]

\[ v_{\text{anal1}}(t) := \frac{1 + \exp\left(-2 \left(\frac{1}{2} \frac{\ln(-1)}{k} \cdot k + 2 \cdot g \cdot t \cdot k\right)\right)}{k - \exp\left(-2 \left(\frac{1}{2} \frac{\ln(-1)}{k} \cdot k + 2 \cdot g \cdot t \cdot k\right)\right)} \]

Now lets plot the results:

![Comparison of Solutions](image)

Lo and behold! They do! Furthermore because of the software, this approach is generally faster than attempting to manipulate analytical expressions.
O. K. - sit back,...take a deep breath, (smoke'em if ya' got'em), assume the details above are correct and look at the big picture. What have we got. We started with a differential equation for acceleration. We ended up with 2 versions of an analytical solution to that differential equation which is itself an equation, relating the variables in the derivative, v and t. We can now compute the velocity of the particle at ANY time t by simply plugging a t value into either of our equations and solving for v (we better get the same answer !). The KEY to doing this hinged on being able to solve the original differential equation. Suppose we could NOT solve it analytically (in most real world situations we can't) ? What do we do ?

Well, it turns out there is another, totally different, way to solve the problem which does not rely on calculus BUT only gives us an approximate solution. The technique is one of a number of procedures called "numerical methods". What we do is approximate the original differential equation as follows:

\[ \frac{dv}{dt} = \frac{Δv}{Δt} = g \left(1 - k^2 \cdot v^2\right) \]

That is, we let the differentials of \( v \) and \( t \) take on real values, \( Δt \) and \( Δv \). This is called a "finite difference approximation". We can then expand \( Δv \) and \( Δt \) into: \( Δt = t_2 - t_1 \) and \( Δv = v_2 - v_1 \) solve for \( v_2 \) and \( t_2 \) to get \( t_2 = t_1 + Δt \) and \( v_2 = v_1 + Δv \)

Summary so far : Here are the equations we have developed

1. \( \frac{Δv}{Δt} = g \left(1 - k^2 \cdot v^2\right) \) \( \implies \) \( v_{i+1} - v_i = g \left[1 - k^2 \cdot (v_i)^2\right] \cdot (t_{i+1} - t_i) \)

It should be obvious that we can also write the following equations:

2. \( v_{i+1} = v_i + Δv_i \)
3. \( t_{i+1} = t_i + Δt_i \)
4. \( x_{i+1} = x_i + v_i \cdot Δt \)

Close inspection indicates that we have an explicit, recursive solution procedure here if: (1.) We know the velocity at a point in time which gives us a place to start the solution process. Recall the we needed this information in the analytical procedure also, to evaluate the constant of integration. Well...we know that \( v = 0 \) ft/sec when \( t = 0 \) sec so \( v_0 = 0 \) when \( t_0 = 0 \). and (2.) We know \( Δt \). Without getting into alot of theory here let's just say that the choice of the "time step", \( Δt \), is up to the user but should be small. Now we can proceed.
Explicit, Recursive Solution Procedure

1. Compute \( v_{i+1} \) from eqn. 1 based on starting values of \( v \) and \( t \) (both zero) and a chosen value of \( \Delta t \).

2. Once computed, replace \( v_i \) with \( v_{i+1} \) and \( t_i \) with \( t_{i+1} \). Note that if \( \Delta t \) is constant we don’t really don’t have to compute the individual time values.

3. repeat the procedure, each time replacing "i" values with "i+1" values

4. Once the counter, \( i \), hits a specified value, stop

Let's do it with numbers!

\[ n := 100 \quad \text{number of time increments, value can be easily changed, counter will stop when we get to "n"} \]

\[ i := 0, 1 \ldots n \quad \text{counter for i} \]

\[
\begin{pmatrix}
  v_0 \\
  \Delta v_0 \\
  x_0
\end{pmatrix}
:=
\begin{pmatrix}
  0 \text{ m} \\
  0 \text{ m} \\
  0 \text{ m}
\end{pmatrix}
\]

Initial conditions for each parameter. We define all three at once using vectors

\[ \Delta t := 0.15 \text{ sec} \quad \text{Size of time step in seconds, easily changed. In this example we made it relatively large so that the 2 curves plotted below would be distinguishable.} \]

\[ \text{time}_i := i \cdot \Delta t \quad \text{total elapsed time in seconds} \]

\[ k := \frac{0.1 \text{ sec}}{\text{m}} \]

Equations to be solved. We use vectors here simply for compactness and convenience. Mathcad cycles through all three equations for each value of \( i \), stopping when \( i = 100 \)

\[
\begin{pmatrix}
  \Delta v_{i+1} \\
  v_{i+1} \\
  x_{i+1}
\end{pmatrix}
:=
\begin{pmatrix}
  g \left[ 1 - k^2 (v_i)^2 \right] \Delta t \\
  v_i + \Delta v_i \\
  x_i + v_i \Delta t
\end{pmatrix}
\]
GRAPHICALLY COMPARE THE NUMERICAL AND ANALYTICAL SOLUTIONS

Let's graphically compare the numerical solution for velocity vs time to the analytical solution developed earlier.

Plotting variable for the analytical solution: \( \Delta t := .01 \text{ sec} \), \(.09 \text{ sec}\), \(3 \text{ sec}\)

\[
v_{\text{num}}(\Delta t) := \left[ 1 + \exp\left( -2 \left( \frac{-1}{2} \cdot \frac{\ln(-1)}{k} \right) k + 2 g \Delta t k \right) \right]
\]

Now plot \( v \) vs \( t \) from each technique, 140 total points. We deliberately choose a "large" \( \Delta t \) so the graphs would not plot on top of each other. As \( \Delta t \) is made smaller the 2 curves begin to coincide. You try it with \( \Delta t = 0.05 \text{ sec} \).

Comparison of solutions

\[\text{falling_particle_with_resistance_prob_11-21b.mcd 7 3/23/00}\]
So what are differences in the analytical and numerical methods? We'll list some below

1. The analytical method requires calculus.

2. Aside from the finite difference approximation the numerical procedure doesn't require calculus but does require ALOT of simple calculations. In fact you really can't do numerical methods without a computer.

3. The analytical method results in an equation which we can use to compute the velocity at any time t simply by plugging in the desired t value. Conversely the numerical method results in a set of values, not an equation. However, if we've done everything right the values should match those obtained from the analytical solution.

4. Unlike the analytical solution, the numerical solution requires that we ALWAYS start at the beginning and "step through" the solution procedure until we get to the desired t value.

5. If \( \Delta t \) is chosen "too big" the finite difference approximation breaks down. However, the 2 techniques can produce results as close to each other as needed if \( \Delta t \) is chosen "small enough".

The user can "play around" with the values of k, n, and \( \Delta t \) to gain some perspective on the physical meaning of k or to see why it is necessary to choose n and \( \Delta t \) properly.

Possible questions for discussion

1. What physical parameters affect the value of k?

2. How does the choice of the size or number of time steps affect the numerical solution? Is the solution technique used implicit or explicit?

3. Under what conditions would a similar problem require a numerical solution? For this problem how did you test the correctness of the numerical solution?

4. Could you devise a procedure, based on the equation given, for actually measuring the value of k?

5. Under what conditions is the vector sum of the forces on an object equal to zero?