There are 2 basic mathematical approaches to the solution of dynamics problems in particular and mathematics problems in general. Analytical and numerical. An analytical solution is obtained by following the rules of calculus and obtaining an equation into which values can be "plugged" to get an answer. A numerical solution involves either "guessing" at a solution or, as in the case below, approximating the derivative and performing many simple arithmetic calculations over time to arrive at an answer. Because many calculations are involved numerical solutions are often characterized as requiring a computer. Numerical solutions are often required where analytical solutions are not possible.

"WAR STORY"

During WWII artillery pieces could not be used (or even shipped) unless accompanied by firing tables which told the artillarymen how to aim the gun given such things as elevation, wind speed, air temperature etc. The computation of firing tables was simply numerical solutions of the equations for a trajectory computed for a variety of conditions. Since there were no computers in WWII firing tables were computed manually by women or groups of women (men were at war). The women were called ...you guessed it..."computers". Nowadays all these factors are entered into a real computer which automatically computes the firing angle of the gun.

Example Problem

A projectile is fired with an initial velocity of 100 m/sec at an angle of 35\(^\circ\). Determine the trajectory and angle with the horizontal plane over time

**Initial conditions**

... Initial velocity and firing angle \(v_0 := 500 \frac{m}{sec}\) and \(\theta_0 := 36\) deg

initial x and y components of the velocity \(v_x := v_0 \cdot \cos(\theta_0)\) and \(v_y := v_0 \cdot \sin(\theta_0)\)

Fire shell from the origin \(x_0 := 0\) m and \(y_0 := 0\) m

Assume shell has a constant velocity coming out of the barrel, acceleration in the x direction is zero, acceleration in the y direction is \(-g\)

Acceleration of gravity: \(a := -g\)
NUMERICAL SOLUTION - Finite Difference Approximation

\[ i := 1 \ldots 5000 \text{ counter} \]

\[ \Delta t := 0.02 \text{ sec} \quad \ldots \text{Time divided into discrete intervals} \]

Develop finite difference approximation form of the governing differential equation \( a = \frac{d}{dt}v \).

If we let velocity and time take on finite changes we can write:

\[ \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} = a \]

Rewrite the last 2 terms as:

\[ v_2 = v_1 + a(t_2 - t_1) = v_1 + a \cdot (\Delta t) \]

If we let \( \Delta t \) stand for a time interval rather than a specific time we can write:

\[ v_{\text{new}} = v_{\text{old}} + a \cdot \Delta t \quad \text{This is the form of the equation used below} \]

\[ v_{x_i} := v_{x_0} \quad \text{In the absence of resistance the velocity on the x direction is constant} \]

\[ v_{y_i} := v_{y_{i-1}} + a \cdot \Delta t \quad \text{velocity in the y direction affected by gravity} \]

Now we can integrate the expressions for velocity, \( v = \frac{d}{dt}x \) to get position:

\[ x_{i+1} := x_i + v_{x_i} \cdot \Delta t \quad \text{x coordinate vs time} \]

\[ y_{i+1} := y_i + v_{y_i} \cdot \Delta t + \frac{a}{2} \cdot (\Delta t)^2 \quad \text{y coordinate vs time} \]

Notice that the velocity in the y direction would be characterized as "uniformly accelerated" since g is constant.

\[ \theta_i := \left( \text{atan} \left( \frac{v_{y_i}}{v_{x_i}} \right) \right) \ldots \text{direction of shell at any time} \]
For a firing angle of 35° the distance traveled 14.95 mi. while for a firing angle of 36° the distance traveled is 15.11 miles (15.11 mi – 14.95 mi = 844.8 ft). Thus, a change of 1 degree causes the projectile to miss the intended target by nearly 850 ft. (almost 3 football fields). The reality is that it is not as easy as it looks to hit something that is reasonably far away. Keep in mind this solution doesn’t take into account wind resistance, temperature, elevation, humidity, wind speed, or curvature of the earth. It also assumes the target and gun aren’t moving.
A graphical approach to determining time of flight.

We know that the shell has a y coordinate of zero at \( t = 0.0 \) sec and when the shell hits the target. Below we plot the y coordinate vs time and read off the time the shell hits the target using the trace function.

\[
\frac{y_{i+1}}{y_d} \cdot \text{sec}
\]

\text{time of flight} = 59.8 \text{ sec.}